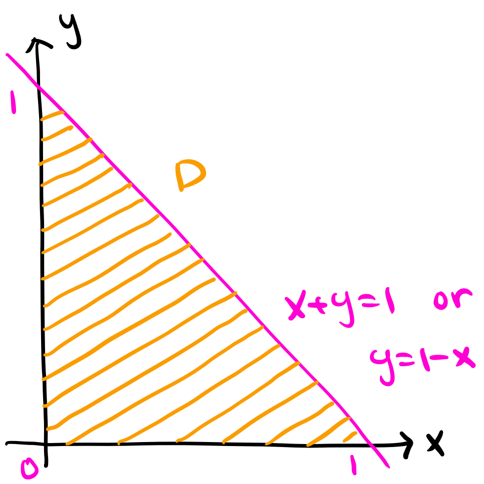
1. Consider the double integral

$$\int\int_D xy\, dA$$

over the triangular region D bounded by the three straight lines $x = 0$, $y = 0$, and $x + y = 1$.

(a) (2 points) What are the three vertices of D?



$x+y=1$ is a line with

x-intercept $=1$, y-intercept $=1$

The vertices of D are $\boxed{(0,0),\ (1,0),\text{ and } (0,1)}$

(b) (8 points) Evaluate the integral.

D is given by $0 \le x \le 1$, $0 \le y \le 1-x$.

$$\iint_D xy\,dA = \int_0^1 \int_0^{1-x} xy\,dy\,dx = \int_0^1 \left.\frac{xy^2}{2}\right|_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \frac{1}{2}x(1-x)^2\,dx = \frac{1}{2}\int_0^1 x - 2x^2 + x^3\,dx$$

$$= \frac{1}{2}\left(\frac{x^2}{2} - \frac{2}{3}x^3 + \frac{x^4}{4}\right)\Bigg|_{x=0}^{x=1} = \boxed{\frac{1}{24}}$$

<u>Note</u> You can also describe D by $0 \le y \le 1$, $0 \le x \le 1-y$.

2. Let $f(x,y) = \frac{y^2}{2} - \cos x$.

(a) (5 points) Find all the critical points of $f(x,y)$ satisfying $-\frac{3\pi}{2} < x < \frac{3\pi}{2}$.

$$\nabla g = (g_x, g_y) = (\sin x, y)$$

$$\nabla g = (0,0) \Rightarrow \begin{cases} \sin x = 0 \rightsquigarrow x = 0, \pm\pi . \\ y = 0 \end{cases}$$

The critical points are $\boxed{(0,0) \text{ and } (\pm\pi, 0)}$

(b) (5 points) Classify the critical points of (a) as maxima, minima, or saddles.

The Hessian of $g(x,y)$ is

$$H = \det \begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix} = g_{xx} \cdot g_{yy} - g_{xy}^2 .$$

$$g_{xx} = \frac{\partial g_x}{\partial x} = \frac{\partial}{\partial x}(\sin x) = \cos x .$$

$$g_{xy} = \frac{\partial g_x}{\partial y} = \frac{\partial}{\partial y}(\sin x) = 0 .$$

$$g_{xy} = \frac{\partial g_y}{\partial y} = \frac{\partial}{\partial y}(y) = 1 .$$

At $(0,0)$: $H = 1 \cdot 1 - 0^2 = 1 > 0$, $g_{xx} = 1 > 0$

$\rightsquigarrow$ a local minimum.

At $(\pm\pi, 0)$: $H = (-1) \cdot 1 - 0^2 = -1 < 0$.

$\rightsquigarrow$ saddle points.

$$\Rightarrow \boxed{\begin{cases} \text{a local minimum at } (0,0) \\ \text{saddle points at } (\pm\pi, 0) \end{cases}}$$

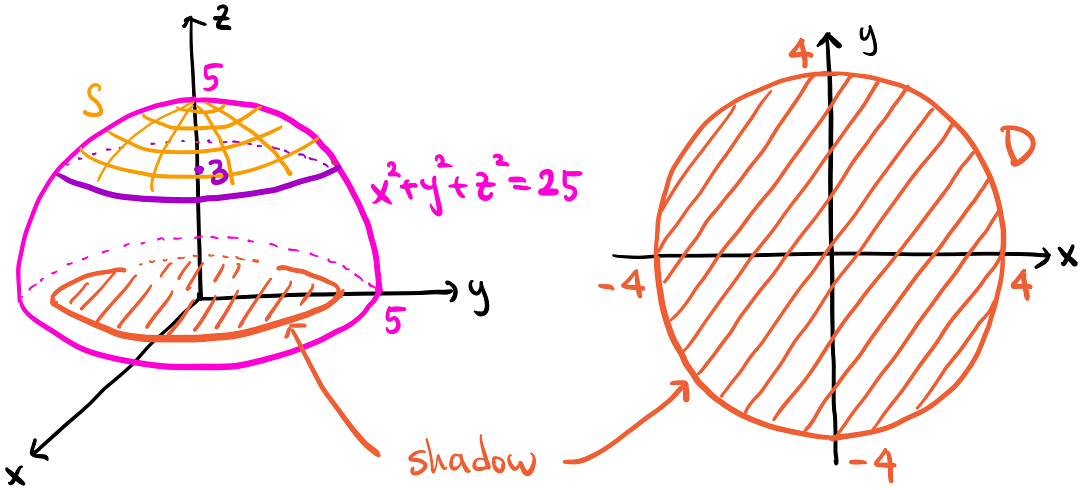
3. Find the area of the spherical surface $x^2 + y^2 + z^2 = 25$ above the plane $z = 3$ in the following steps:

(a) (3 points) Find the partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ using implicit differentiation.

The sphere is a level surface of $f(x,y,z) = x^2+y^2+z^2$.

$$\frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = -\frac{2x}{2z} = \boxed{-\frac{x}{z}}, \quad \frac{\partial z}{\partial y} = -\frac{f_y}{f_z} = -\frac{2y}{2z} = \boxed{-\frac{y}{z}}$$

(b) (3 points) Set up a double integral for the surface area using the formula $\int\int_D \sqrt{1 + \frac{\partial z}{\partial x}^2 + \frac{\partial z}{\partial y}^2}\, dA$.



$z = 3 \Rightarrow x^2+y^2 = 25 - z^2 = 25 - 3^2 = 16$.

The shadow D on the xy-plane is given by $x^2+y^2 \leq 16$

In polar coordinates, D is given by $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 4$.

$$\text{Area} = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA \underset{\text{(a)}}{=} \iint_D \sqrt{\frac{x^2+y^2+z^2}{z^2}}\, dA$$

$$\underset{x^2+y^2+z^2=25}{=} \iint_D \sqrt{\frac{25}{25-x^2-y^2}}\, dA = \boxed{\int_0^{2\pi}\int_0^4 \frac{5}{\sqrt{25-r^2}} \cdot r\, dr\, d\theta}$$

Jacobian

(c) (4 points) Find the area.

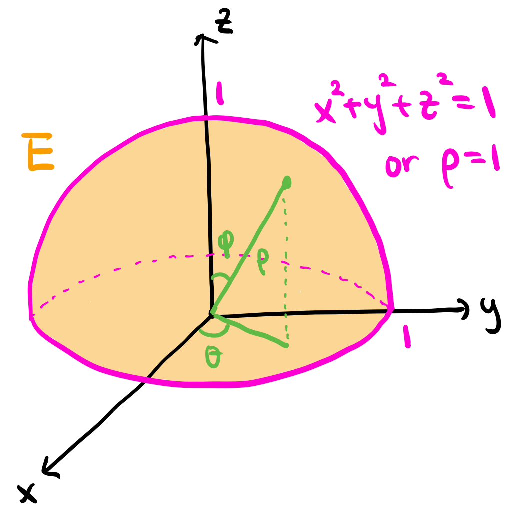
$$\text{Area} = \int_0^{2\pi}\int_0^4 \frac{5}{\sqrt{25-r^2}} \cdot r\, dr\, d\theta \underset{u=25-r^2}{=} \int_0^{2\pi}\int_{25}^{9} \frac{5}{\sqrt{u}} \cdot \left(-\frac{1}{2}\right) du\, d\theta$$

$$= \int_0^{2\pi} -5\sqrt{u}\,\Big|_{u=25}^{u=9}\, d\theta = \int_0^{2\pi} 10\, d\theta = \boxed{20\pi}$$

4. Let E be the solid hemisphere $0 \le x^2 + y^2 + z^2 \le 1$ with $z \ge 0$.

(a) (5 points) Evaluate the volume integral

$$\iiint_E z\, dV.$$



In spherical coordinates:

$x^2+y^2+z^2=1 \rightsquigarrow \rho^2=1 \rightsquigarrow \rho=1$

φ is maximized on the xy-plane

$\Rightarrow$ The solid E is given by

$0 \le \theta \le 2\pi,\ 0 \le \varphi \le \frac{\pi}{2},\ 0 \le \rho \le 1$

$$\iiint_E z\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho\cos\varphi \cdot \underbrace{\rho^2\sin\varphi}_{\text{Jacobian}}\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{\rho^4}{4}\cos\varphi\sin\varphi \Big|_{\rho=0}^{\rho=1} d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{1}{4}\cos\varphi\sin\varphi\, d\varphi\, d\theta \underset{u=\sin\theta}{=} \int_0^{2\pi}\int_0^1 \frac{1}{4} u\, du\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{8}u^2\Big|_{u=0}^{u=1} d\theta = \int_0^{2\pi}\frac{1}{8}\, d\theta = \boxed{\frac{\pi}{4}}$$

(b) (5 points) Evaluate the volume integral

$$\iiint_E (x^2+y^2+z^2)^{1/2}\, dV.$$

$$\iiint_E \sqrt{x^2+y^2+z^2}\, dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 \rho\cdot\underbrace{\rho^2\sin\varphi}_{\text{Jacobian}}\, d\rho\, d\varphi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2}\frac{\rho^4}{4}\sin\varphi\Big|_{\rho=0}^{\rho=1} d\varphi\, d\theta = \int_0^{2\pi}\int_0^{\pi/2}\frac{1}{4}\sin\varphi\, d\varphi\, d\theta$$

$$= \int_0^{2\pi} -\frac{1}{4}\cos\varphi\Big|_{\varphi=0}^{\varphi=\pi/2} d\theta = \int_0^{2\pi}\frac{1}{4}\, d\theta = \boxed{\frac{\pi}{2}}$$

5. (10 points) Find the maximum and minimum values of the function $f(x, y) = x + y$ on the curve $x^2 + y^2 - xy = 4$.

Set $g(x,y) = x^2+y^2-xy-4$.

Solve $\nabla f = \lambda \nabla g$ and $g=0$

$\Rightarrow (1,1) = \lambda(2x-y, 2y-x)$ and $x^2+y^2-xy-4=0$

$\rightsquigarrow 1 = \lambda(2x-y),\ 1=\lambda(2y-x),\ x^2+y^2-xy=4.$

$\rightsquigarrow 2x-y = \frac{1}{\lambda},\ 2y-x = \frac{1}{\lambda},\ x^2+y^2-xy=4$

$\Rightarrow 2x-y = 2y-x \Rightarrow x=y.$

$x^2+y^2-xy=4 \rightsquigarrow x^2+x^2-x^2=4 \rightsquigarrow x=\pm 2$

$\Rightarrow (x,y) = (-2,-2)$ or $(2,2)$.

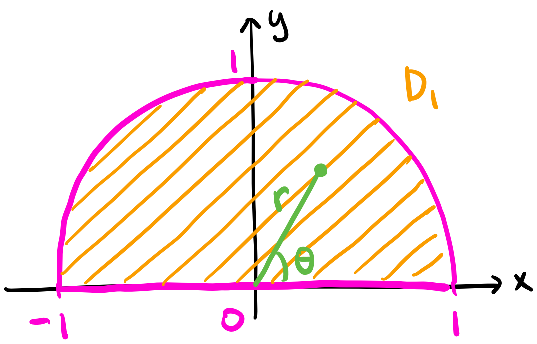
$f(-2,-2) = -2-2=-4,\ f(2,2) = 2+2=4.$

$$\Rightarrow \boxed{\begin{cases} \text{Maximum} = 4 \text{ at } (2,2) \\ \text{Minimum} = -4 \text{ at } (-2,-2) \end{cases}}$$

<u>Note</u> For this problem, you can't remove the constraint $x^2+y^2-xy=4$, because you can't express x or y as a function of the other variable.

6. This problem is about finding the y coordinate of the center of mass.

(a) (3 points) Consider the half disc $0 \le x^2+y^2 \le 1$ with $y \ge 0$. Assume that the density is $\rho(x,y) = 1$. Find $\bar{y}$, the y-coordinate of the center of mass of the half-disc.



In polar coordinates, the domain D_1 is given by $0 \le \theta \le \pi$, $0 \le r \le 1$.

$$m_{D_1} = \iint_{D_1} \rho(x,y)\,dA = \iint_{D_1} 1\,dA = \text{Area}(D_1) = \frac{1}{2}\underbrace{\pi \cdot 1^2}_{\text{Area of circle}} = \frac{\pi}{2}$$

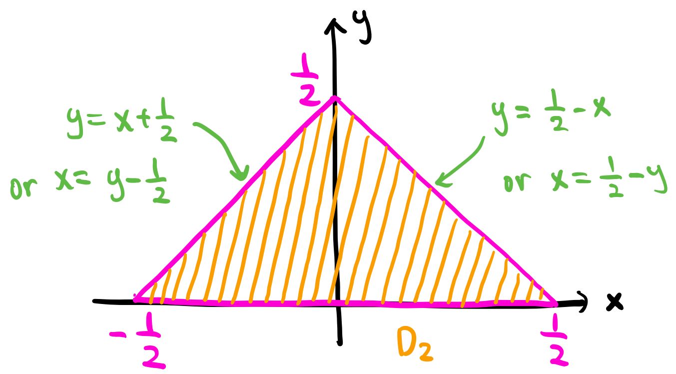
$$\bar{y}_{D_1} = \frac{1}{m_{D_1}}\iint_{D_1} y\rho(x,y)\,dA = \frac{2}{\pi}\iint_{D_1} y\,dA$$

Jacobian

$$= \frac{2}{\pi}\int_0^{\pi}\int_0^1 r\sin\theta \cdot r\,dr\,d\theta = \frac{2}{\pi}\int_0^{\pi} \frac{r^3}{3}\sin\theta \Big|_{r=0}^{r=1} d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{1}{3}\sin\theta\,d\theta = \frac{2}{\pi}\cdot\left(-\frac{1}{3}\cos\theta\right)\Big|_{\theta=0}^{\theta=\pi} = \boxed{\frac{4}{3\pi}}$$

(b) (3 points) Find the y-coordinate of the center of mass of the triangular region with vertices at $(-1/2, 0)$, $(1/2, 0)$, and $(0, 1/2)$ assuming density $\rho(x,y) = 1$.



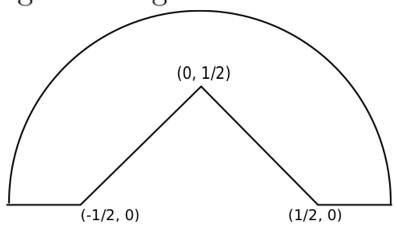
The domain D_2 is given by $0 \le y \le \frac{1}{2}$, $y - \frac{1}{2} \le x \le \frac{1}{2} - y$.

$$m_{D_2} = \iint_{D_2} \rho(x,y)\,dA = \iint_{D_2} 1\,dA = \text{Area}(D_2) = \frac{1}{2}\cdot 1 \cdot \frac{1}{2} = \frac{1}{4}.$$

$$\bar{y}_{D_2} = \frac{1}{m_{D_2}}\iint_{D_2} y\rho(x,y)\,dA = 4\iint_{D_2} y\,dA = 4\int_0^{\frac{1}{2}}\int_{y-\frac{1}{2}}^{\frac{1}{2}-y} y\,dx\,dy$$

$$= 4\int_0^{\frac{1}{2}} (1-2y)\,y\,dy = 4\int_0^{\frac{1}{2}} y - 2y^2\,dy = 4\left(\frac{y^2}{2} - \frac{2}{3}y^3\right)\Big|_{y=0}^{y=\frac{1}{2}} = \boxed{\frac{1}{6}}$$

(c) (4 points) Now suppose a triangular wedge is removed from the half disc to get the following region:



The density is again $\rho(x, y) = 1$. Find $\bar{y}$, the y-coordinate of the center of mass of this region.

Let D be the given domain $\Rightarrow D \cup D_2 = D_1$.

$$m_D = m_{D_1} - m_{D_2} \underset{\substack{\uparrow \\ (a),(b)}}{=} \frac{\pi}{2} - \frac{1}{4} = \frac{2\pi - 1}{4}$$

$$\bar{y}_D = \frac{1}{m_D} \iint_D y\rho(x,y)\,dA = \frac{4}{2\pi - 1} \iint_D y\,dA$$

$$= \frac{4}{2\pi - 1}\left(\iint_{D_1} y\,dA - \iint_{D_2} y\,dA\right).$$

$$\iint_{D_1} y\,dA = m_{D_1} \cdot \bar{y}_{D_1} \underset{\substack{\uparrow \\ (a)}}{=} \frac{\pi}{2} \cdot \frac{4}{3\pi} = \frac{2}{3}.$$

$$\iint_{D_2} y\,dA = m_{D_2} \cdot \bar{y}_{D_2} \underset{\substack{\uparrow \\ (b)}}{=} \frac{1}{4} \cdot \frac{1}{6} = \frac{1}{24}.$$

$$\Rightarrow \bar{y}_D = \frac{4}{2\pi - 1}\left(\frac{2}{3} - \frac{1}{24}\right) = \boxed{\frac{5}{2(2\pi - 1)}}$$

<u>Note</u> It's possible to set up the integrals in polar coordinates.

$$y = \frac{1}{2} - x \rightsquigarrow x + y = \frac{1}{2} \rightsquigarrow r(\cos\theta + \sin\theta) = \frac{1}{2} \rightsquigarrow r = \frac{1}{2(\cos\theta + \sin\theta)}$$

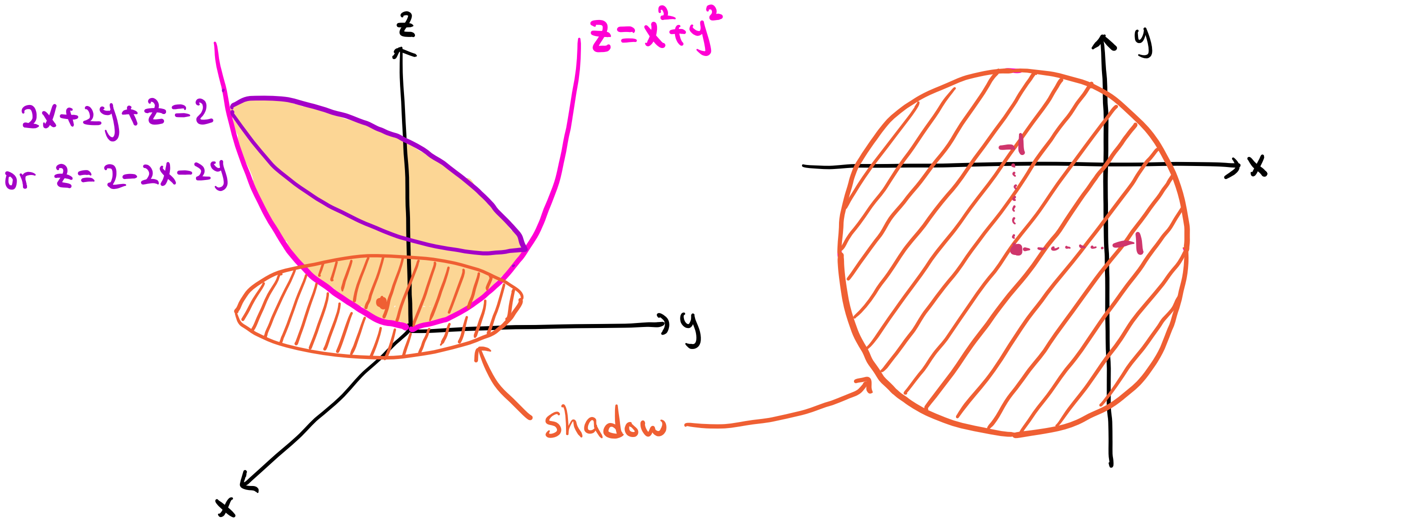
$$y = \frac{1}{2} + x \rightsquigarrow y - x = \frac{1}{2} \rightsquigarrow r(\sin\theta - \cos\theta) = \frac{1}{2} \rightsquigarrow r = \frac{1}{2(\sin\theta - \cos\theta)}$$

$$\Rightarrow \begin{cases} 0 \le \theta \le \frac{\pi}{2}, & 0 \le r \le \frac{1}{2(\cos\theta + \sin\theta)} \\ \frac{\pi}{2} \le \theta \le \pi, & 0 \le r \le \frac{1}{2(\sin\theta - \cos\theta)} \end{cases}$$

7. Consider the paraboloid $z = x^2 + y^2$ and the plane $2x + 2y + z = 2$.

Very tricky!

(a) (2 points) Approximately sketch the volume bounded by the paraboloid and the plane. The plane is above the volume and the paraboloid is below it.



(b) (4 points) Express the volume as a double integral over a region in the x-y plane.

Intersection: $z = x^2+y^2$ and $z = 2-2x-2y$

$\Rightarrow x^2+y^2 = 2-2x-2y \Rightarrow x^2+y^2+2x+2y+2 = 4$

$\Rightarrow (x+1)^2 + (y+1)^2 = 4.$

The shadow D on the xy-plane is given by $(x+1)^2+(y+1)^2 \le 4$

The solid is between the surfaces $z = x^2+y^2$ and $z = 2-2x-2y$.

$\Rightarrow$ Volume $= \iint_D (2-2x-2y) - (x^2+y^2)\, dA = \iint_D 4-(x+1)^2-(y+1)^2\, dA$

We use the "shifted" polar coordinates centered at $(-1,-1)$

$\Rightarrow x = -1 + r\cos\theta, \quad y = -1 + r\sin\theta$

D is given by $0 \le \theta \le 2\pi, \quad 0 \le r \le 2.$

Jacobian

$\Rightarrow$ Volume $= \boxed{\int_0^{2\pi}\int_0^{2} (4-r^2)\, r\, dr\, d\theta}$

(c) (4 points) Find the volume of the region bounded by the paraboloid and the plane.

Volume $= \int_0^{2\pi}\int_0^{2} 4r - r^3\, dr\, d\theta = \int_0^{2\pi} 2r^2 - \frac{r^4}{4}\Big|_{r=0}^{r=2} d\theta = \int_0^{2\pi} 4\, d\theta = \boxed{8\pi}$